Singular solutions of the Stokes problems in a power cusp domain

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2018-12-11

If we denote by $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, some domain and by $[0, T)$ a fixed time interval with $0 < T \leq \infty$, then the full nonlinear Navier-Stokes system in $\Omega \times [0, T)$ has the form

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p & = & \mathbf{f}, & x \in \Omega, \\ \operatorname{div}\mathbf{u} & = & 0, & x \in \Omega, \\ \mathbf{u}|_{\partial\Omega} & = & \mathbf{a}, & x \in \partial\Omega, \\ \mathbf{u}(x, 0) & = & \mathbf{u}_0(x), & x \in \Omega. \end{cases}$$

Here $\mathbf{u} = \mathbf{u}(x, t) = (\mathbf{u}_1(x, t), ..., \mathbf{u}_n(x, t))$ is the unknown velocity, $p = p(x, t)$ is the unknown pressure, $\mathbf{f} = \mathbf{f}(x, t) = (\mathbf{f}_1(x, t), ..., \mathbf{f}_n(x, t))$ the given density of the exterior force, $\nu > 0$ the viscosity coefficient, $\mathbf{a} = \mathbf{a}(x, t) = (\mathbf{a}_1(x, t), ..., \mathbf{a}_n(x, t))$ the given boundary value and $\mathbf{u}_0(x)$ the given initial velocity with $t \in [0, T)$, $x = (x_1, ..., x_n) \in \Omega$.

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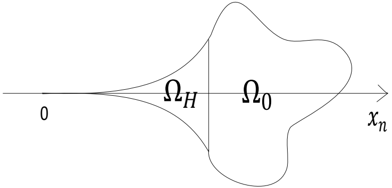
Domain $\Omega = \Omega_H \cup \Omega_0$



Where

$$\Omega_H = \left\{x \in \mathbf{R}^n : \ |x'| < \varphi(x_n), \ x_n \in (0, H)\right\},$$

$n = 2{,}3$, $\varphi(x_n) = \gamma_0 \cdot x_n^\lambda$, $\gamma_0 = \text{const}$, $\lambda > 1$, $\partial\Omega_0$ is Lipschitz and $\sigma(h) = \{x \in \Omega_H : x_n = h = \text{const}\}$, is the cross section of Ω_H by the plane $x_n = h$, $x' = (x_1, ..., x_{n-1})$.

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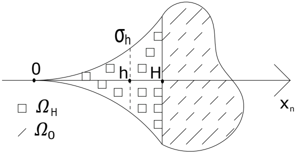

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Constructing the asymptotic representation of the solution to the Stokes problems near the cusp point O we use the ideas proposed in the paper

- *S.A. Nazarov, K. Pileckas, Asymptotics of Solutions to Stokes and Navier-Stokes equations in domains with paraboloidal outlets to infinity, Rend. Sem. Math. Univ. Padova,* **99** *(1998), 1–43,*

where the asymptotic behaviour of the solutions to stationary Stokes and Navier-Stokes problems was studied in unbounded domains with paraboloidal outlets to infinity. In turn the method used in • is a variant of the algorithm of constructing the asymptotics for solutions to elliptic equations in slender domains, see, e.g.,...

…,see, e.g. for arbitrary elliptic problems

- *V.G. Maz'ya, S.A. Nazarov, B.A. Plamenevskij, Asymptotics of the solution of the Dirichlet problem in thin domains, singularly perturbed domains, (1982-2000);*

for the stationary Stokes and Navier–Stokes equations

- *S.A. Nazarov, K. Pileckas, Asymptotic solution of the Navier–Stokes problem in a thin channel,(1990);*

for the nonstationary and time-periodic Navier–Stokes equations

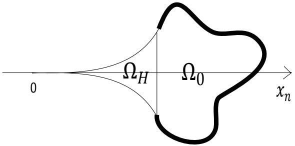
- *G. Panasenko, K. Pileckas, Asymptotic analysis of the non-steady Navier-Stokes equations in a thin pipe, tube structure (2012-now).*

We consider the steady Stokes problem:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & x \in \Omega, \\ \operatorname{div}\mathbf{u} = 0, & x \in \Omega, \\ \mathbf{u} = \mathbf{a}, & x \in \partial\Omega. \end{cases} \tag{1}$$

$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,ds = F \not\equiv 0, \tag{2}$$

where $\operatorname{supp}\mathbf{a} \in \partial\Omega_0 \cap \partial\Omega$,



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where $\operatorname{supp}\mathbf{a} \in \partial\Omega_0 \cap \partial\Omega$, $\nu > 0$ is the constant kinematic viscosity. The necessary compatibility condition

$$F + \int\limits_{\partial\Omega\cap\partial\Omega_0} \mathbf{a}(x)\cdot\mathbf{n}\,dx = 0.$$

Definition 1

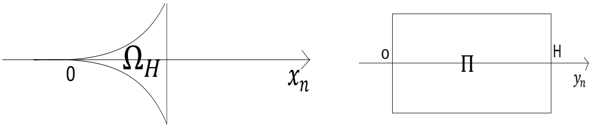
Let $\mathbf{f} \in L^2(\Omega)$ and $\mathbf{a} \in W^{1/2,2}(\partial\Omega_{H/2,H})$ be given functions. By a weak solution of problem (1) we understand a solenoidal vector field $\mathbf{u} \in W^{1,2}_{loc}(\Omega)$ satisfying the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{a}$, the flux condition (2) and the integral identity

$$\nu \int\limits_{\Omega} \nabla \mathbf{u} \cdot \nabla \boldsymbol{\eta} \, dx = \int\limits_{\Omega} \mathbf{f} \cdot \boldsymbol{\eta} \, dx, \ \forall \ \boldsymbol{\eta} \in C_0^\infty(\Omega), \ \ \mathrm{div} \boldsymbol{\eta} = 0.$$

Let us consider the homogeneous problem (1), (2) with zero boundary condition in the domain Ω_H.

$$(x', x_n) \longrightarrow \left(\frac{x'}{x_n^{\lambda}}, x_n \right) := (y', y_n).$$

$$\Omega_H \to \Pi$$



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$$(x', x_n) \longrightarrow \left(\frac{x'}{x_n^{\lambda}}, x_n\right) := (y', y_n).$$

$$\begin{cases} -\nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p = 0, \quad y \in \Pi, \\ -\nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D}p = 0, \quad y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D}\, u_n = 0, \quad y \in \Pi, \\ \mathbf{u}|_{\partial\Pi} = 0, \end{cases} \tag{3}$$

where $\Delta' = \nabla' \cdot \nabla'$, $\nabla' = (\partial_1, \ldots, \partial_{n-1})$, $\mathfrak{D} = \partial_n - \lambda y_n^{-1} y' \cdot \nabla'$, $\mathbf{u}' = (u_1, \ldots, u_{n-1})$, $\partial_k = \frac{\partial}{\partial y_k}$.

We look for the solution $(\mathbf{U}_0, P_0)$ in the form

$$P_0(y', y_n) = q_0(y_n) + Q_0(y', y_n),$$
$$\mathbf{U}_0(y', y_n, t) = (\mathbf{U}_0'(y', y_n), U_{n,0}(y', y_n))$$

with

$$U_{n,0}(y', y_n) = y_n^{2\lambda} \partial_n q_0(y_n) \varphi(y').$$

($(\mathbf{U}_0, P_0)$ is not the exact solution!)

Substituting $(\mathbf{U}_0, P_0)$ into equations as $y_n \to 0$, we get

$$\begin{cases} -\nu\Delta'\mathbf{U}_0' + \nabla' y_n^\lambda Q_0 = 0, \quad y' \in \omega, \\ \mathrm{div}'\mathbf{U}_0' = G_0(y', y_n), \quad y' \in \omega, \\ \mathbf{U}_0'|_{\partial\omega} = 0, \end{cases} \qquad (*)$$

$$\begin{cases} \nu\Delta'\varphi(y') = 1, \quad y' \in \omega, \\ \varphi(y')|_{\partial\omega} = 0, \end{cases}$$

where $\omega = \left\{ y' \in \mathbf{R}^{n-1} : \, |y'| < \gamma_0 \right\}$.

$$G_0(y', y_n) = -y_n^\lambda \mathfrak{D}(y_n^{2\lambda}\partial_n q_0(y_n)\varphi(y')).$$

$$\begin{cases} \nu\Delta'\varphi(y') = 1, & y' \in \omega, \\ \varphi(y')|_{\partial\omega} = 0, \end{cases}$$

The solution $\varphi(y')$ has the form

$$\varphi(y') = \frac{1}{2\nu(n-1)}\left(|y'|^2 - \gamma_0^2\right).$$

$$\begin{cases} -\nu\Delta'\mathbf{U}_0' + \nabla' y_n^\lambda Q_0 = 0, \quad y' \in \omega, \\ \operatorname{div}'\mathbf{U}_0' = G_0(y', y_n), \quad y' \in \omega, \\ \mathbf{U}_0'|_{\partial\omega} = 0, \end{cases} \qquad (*)$$

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$$(*) \overset{\text{is solvable}}{\Longleftrightarrow} \int\limits_\omega G_0 \, dy' = 0.$$

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$$\Rightarrow -y_n^{\lambda} \partial_n \left[y_n^{2\lambda} \partial_n q_0(y_n) \right] - \lambda(n-1) y_n^{3\lambda - 1} \partial_n q_0(y_n) = 0$$

$$\Rightarrow \partial_n \left[y_n^{\lambda(n+1)} \partial_n q_0(y_n) \right] = 0$$

$$\Rightarrow q_0(y_n) = C_1 y_n^{1-\lambda(n+1)} + C_2$$

$$U_{n,0}(y', y_n) = y_n^{2\lambda} \partial_n q_0(y_n) \varphi(y')$$

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$$\overset{C_2=0}{\Rightarrow} U_{n,0}(y', y_n) = C_1[1-\lambda(n+1)] y_n^{-\lambda(n-1)} \varphi(y')$$

Moreover, since

$$\int\limits_{\sigma(h)} \mathbf{u}_0(x) \cdot \mathbf{n}\, dx' = \kappa_0 C_1(1 - \lambda(n+1)),$$

taking $C_1 = F\left[\kappa_0(1 - \lambda(n+1))\right]^{-1}$, we satisfy the flux condition (2); where $\kappa_0 = \int\limits_{\omega} \varphi(y')\, dy' \neq 0$.

Comparing the power exponents of y_n in (*) and G_0 we conclude that functions $\mathbf{U}_0'(y', y_n)$, $Q_0(y', y_n)$ have to be taken in the form

$$\begin{cases} \mathbf{U}_0'(y', y_n) = y_n^{-\lambda(n-2)-1} \mathscr{U}_0'(y'), \\ Q_0(y', y_n) = y_n^{-\lambda(n-1)-1} \mathscr{Q}_0(y'). \end{cases}$$

Therefore, we can rewrite:

$$\begin{cases} -\nu\Delta'\mathscr{U}_0' + \nabla'\mathscr{Q}_0 = 0, \quad y' \in \omega, \\ \operatorname{div}'\mathscr{U}_0' = \mathscr{G}_0(y'), \quad y' \in \omega, \\ \mathscr{U}_0'|_{\partial\omega} = 0, \end{cases}$$

where

$$\mathscr{G}_0(y') = \frac{\lambda}{\kappa_0} FA(y', \nabla')\varphi(y'),$$

the operator A is given by

$$A(y', \nabla') = n - 1 + y' \cdot \nabla'.$$

Finally,

$$\begin{cases} \mathbf{U}_0'\left(\frac{x'}{x_n^\lambda}, x_n\right) = x_n^{-\lambda(n-2)-1}\mathscr{U}_0'\left(\frac{x'}{x_n^\lambda}\right), \\ Q_0\left(\frac{x'}{x_n^\lambda}, x_n\right) = x_n^{-\lambda(n-1)-1}\mathscr{Q}_0\left(\frac{x'}{x_n^\lambda}\right), \end{cases}$$

$$U_{n,0}\left(\frac{x'}{x_n^\lambda}, x_n\right) = \frac{F}{\kappa_0} x_n^{-\lambda(n-1)}\varphi\left(\frac{x'}{x_n^\lambda}\right).$$

Functions $\mathbf{U}_0$, Q_0 solve Stokes problem with the right-hand side $\mathbf{H}_0$ (discrepancies) having the special form

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{H}_0, & x \in \Omega_H, \\ \operatorname{div}\mathbf{u} = 0, & x \in \Omega_H, \\ \mathbf{u} = 0, & x \in \partial\Omega_H. \end{cases} \tag{4}$$

The discrepancies $\mathbf{H}_0'(y', y_n)$, $H_{n,0}(y', y_n)$ left by functions $\mathbf{U}_0$, Q_0 have the form

$$\begin{cases} \mathbf{H}_0'(y', y_n) = \nu\mathfrak{D}^2\mathbf{U}_0'(y', y_n) = y_n^{-\lambda(n-2)-3}\mathscr{F}_0'(y'), \\ H_{n,0}(y', y_n) = \nu\mathfrak{D}^2 U_{n,0}(y', y_n) - \mathfrak{D}\, Q_0(y', y_n) = y_n^{-\lambda(n-1)-2}\mathscr{F}_{n,0}(y'). \end{cases}$$

Consider equations (3) with the right-hand sides having the special form

$$\begin{cases} -\nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p = y_n^{-\lambda(n-2)-3}\mathscr{F}_0'(y'), \quad y \in \Pi, \\ -\nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D} p = y_n^{-\lambda(n-1)-2}\mathscr{F}_{n,0}(y'), \quad y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D} u_n = 0, \quad y \in \Pi, \\ \mathbf{u}|_{\partial\Pi} = 0. \end{cases}$$

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We will satisfy these equations in main, if we put

$$P_1(y', y_n) = C_2 y_n^{-1-\lambda(n-1)} + y_n^{-3-\lambda(n-3)}\mathscr{Q}_1(y'),$$

$$\mathbf{U}_1'(y', y_n) = y_n^{-3-\lambda(n-4)}\mathscr{U}_1'(y'),$$

$$U_{n,1}(y', y_n) = y_n^{-2-\lambda(n-3)}\mathscr{U}_{n,1}(y'),$$

where

$$\mathscr{U}_{n,1}(y') = C_2(-1-\lambda(n-1))\varphi(y') + \mathscr{U}_{n,1}^*(y').$$

$\mathscr{U}^*_{n,1}$ satisfies the equations

$$\begin{cases} -\nu\Delta'\mathscr{U}^*_{n,1} = \mathscr{F}_{n,0}, \quad y' \in \omega, \\ \mathscr{U}^*_{n,1}|_{\partial\omega} = 0, \end{cases}$$

$(\mathscr{U}'_1, \mathscr{Q}_1)$ is the solution to

$$\begin{cases} -\nu\Delta'\mathscr{U}'_1 + \nabla'\mathscr{Q}_1 = \mathscr{F}'_0, \quad y' \in \omega, \\ \mathrm{div}'\mathscr{U}'_1 = [\lambda A(y', \nabla') - 2(\lambda - 1)]\mathscr{U}_{n,1}, \quad y' \in \omega, \\ \mathscr{U}'_1|_{\partial\omega} = 0, \end{cases}$$

and the constant C_2 is uniquely determined from the solvability condition

$$\int\limits_{\omega} \left[\lambda A(y', \nabla') - 2(\lambda - 1)\right] \mathscr{U}_{n,1}(y')\, dy' = 0.$$

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and the constant C_2 is uniquely determined from the solvability condition ,i.e.

$$C_2 = -\frac{1}{\kappa_0\,[1 + \lambda(n-1)]}\int\limits_{\omega} \mathscr{U}^*_{n,1}\,dy'.$$

The discrepancies $\mathbf{H}_1'(y', y_n)$, $H_{n,1}(y', y_n)$ left by functions $\mathbf{U}_1$, P_1 can be written in the form

$$\begin{cases} \mathbf{H}_1'(y', y_n) = & y_n^{-\lambda(n-4)-5} \mathscr{F}_1'(y'), \\ H_{n,1}(y', y_n) = & y_n^{-\lambda(n-3)-4} \mathscr{F}_{n,1}(y'). \end{cases}$$

$$\left(y_n^{\mu+\lambda-1} \mathscr{F}_0', y_n^{\mu} \mathscr{F}_{n,0} \right) \xrightarrow{\mathbf{H}_0 \to \mathbf{H}_1} \left(y_n^{\mu+3(\lambda-1)} \mathscr{F}_1', y_n^{\mu+2(\lambda-1)} \mathscr{F}_{n,1} \right).$$

i.e.

$$\mu \xrightarrow{\mathbf{H}_0 \to \mathbf{H}_1} \mu + 2(\lambda - 1),$$

We shall keep constructing the functions $(\mathbf{U}_k, P_k)$, $k = 1, 2, \ldots$, till the discrepancies $\mathbf{H}'_k$, $H_{n,k}$ will belong to $L^2(\Omega)$:

$$\begin{array}{cccccc} (\mathbf{U}_0, P_0) & (\mathbf{U}_1, P_1) & (\mathbf{U}_2, P_2) & \ldots & (\mathbf{U}_{J^*}, P_{J^*}) & \ldots \\ (\mathbf{H}'_0, H_{n,0}) & (\mathbf{H}'_1, H_{n,1}) & (\mathbf{H}'_2, H_{n,2}) & \ldots & (\mathbf{H}'_{J^*}, H_{n,J^*}) & \ldots \end{array}$$

One can see that, in particular,

$$J^* \geq \min\left\{ J \in \mathbf{N} : J > \frac{1}{4}\left[\frac{n+2}{\lambda - 1} + n + 3\right]\right\}.$$

$$
\begin{cases}
\mathbf{U}'^{[J]}(y', y_n) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y'), \\
U_n^{[J]}(y', y_n) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y'), \\
P^{[J]}(y', y_n) = & \sum_{k=0}^{J} \Big(C_{k+1} y_n^{-\lambda(n+1-2k)+1-2k} (1 \\
& + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\
& + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y') \Big),
\end{cases}
$$

$$\mathscr{U}_{n,k}(y') = C_{k+1}(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathscr{U}^*_{n,k}(y'),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

$$\begin{cases} \mathbf{U}'^{[J]}(y', y_n) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y'), \\ U_n^{[J]}(y', y_n) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y'), \\ P^{[J]}(y', y_n) = & \sum_{k=0}^{J} \Big(C_{k+1} y_n^{-\lambda(n+1-2k)+1-2k} (1 \\ & + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\ & + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y') \Big), \end{cases}$$

$$\mathscr{U}_{n,k}(y') = C_{k+1}(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathscr{U}^*_{n,k}(y'),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

$$\mathscr{U}'_k(y') \quad \mathscr{Q}_k(y')$$

$$\begin{cases} -\nu\Delta'\mathscr{U}'_k + \nabla'\mathscr{Q}_k = \mathscr{F}'_{k-1}, \quad y' \in \omega, \\ \mathrm{div}'\mathscr{U}'_k = [\lambda A(y', \nabla') - 2k(\lambda - 1)]\,\mathscr{U}_{n,k}, \quad y' \in \omega, \\ \mathscr{U}'_k|_{\partial\omega} = 0, \end{cases}$$

$$\mathscr{U}_{n,k}(y') = C_{k+1}(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathscr{U}^*_{n,k}(y'),$$

with

$$\mathscr{F}'_{k-1} = y_n^{(n-2(k+1))\lambda+1+2(k+1)}\nu\mathfrak{D}^2\mathbf{U}'_{k-1}(y', y_n).$$

$$\begin{cases} \mathbf{U}'^{[J]}(y', y_n) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y'), \\ U_n^{[J]}(y', y_n) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y'), \\ P^{[J]}(y', y_n) = & \sum_{k=0}^{J} \Big(C_{k+1} y_n^{-\lambda(n+1-2k)+1-2k} (1 \\ & + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\ & + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y') \Big), \end{cases}$$

$$\mathscr{U}_{n,k}(y') = C_{k+1}(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathscr{U}^*_{n,k}(y'),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

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here $\delta_k^{\bar{k}}$ is Kronecker's delta.

$$\delta_k^{\bar{k}}$$

$$\begin{cases} \mathbf{U}_{\bar{k}}(y', y_n) = \mathbf{U}_k(y', y_n), \\ P_{\bar{k}}(y', y_n) = C_{\bar{k}+1} \ln\, y_n + y_n^{\lambda(2\bar{k}-n+1)-1-2\bar{k}} \mathscr{Q}_{\bar{k}}(y'), \end{cases}$$

$$\begin{cases} \mathbf{U}'^{[J]}(y', y_n) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y'), \\ U_n^{[J]}(y', y_n) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y'), \\ P^{[J]}(y', y_n) = & \sum_{k=0}^{J} \Big(C_{k+1} y_n^{-\lambda(n+1-2k)+1-2k} (1 \\ & + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\ & + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y') \Big), \end{cases}$$

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here $\delta_k^{\bar{k}}$ is Kronecker's delta.

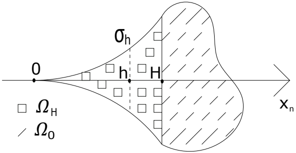
$$\mathscr{U}^*_{n,k}(y')$$

$$\begin{cases} \nu\Delta'\mathscr{U}^*_{n,k} = \mathscr{F}_{n,k-1}, \quad y' \in \omega, \\ \mathscr{U}^*_{n,k}|_{\partial\omega} = 0, \end{cases}$$

with

$$\mathscr{F}_{n,k-1}(y') = y_n^{(n-1-2k)\lambda+2(k+1)}\left(\nu\mathfrak{D}^2 U_{n,k-1}(y', y_n) - \mathscr{Q}_{k-1}(y', y_n)\right).$$

Let us consider the problem (1), (2) in the domain Ω.



Let $\xi \in C^\infty[0,\infty)$ be some nonnegative function

$$\xi(x_n) = \begin{cases} 1, & x_n \leq H/2, \\ 0, & x_n \geq H. \end{cases}$$

Then we can look for the solution $(\mathbf{u}, p)$ to problem (1), (2) in the form

$$\mathbf{u}(x', x_n) = \xi(x_n)\mathbf{U}^{[J^*]}\left(\frac{x'}{x_n^\lambda}, x_n\right) + \mathbf{V}(x', x_n) + \widehat{\mathbf{U}}(x', x_n),$$

$$p(x', x_n) = \xi(x_n)P^{[J^*]}\left(\frac{x'}{x_n^\lambda}, x_n\right) + \widehat{P}(x', x_n),$$

where $(\mathbf{U}^{[J^*]}, P^{[J^*]})$ is the asymptotical decomposition.

We define $\mathbf{V}(x', x_n) = 0$ for $x_n \leq H/2$, while for $x_n \geq H/2$ the function $\mathbf{V}$ is the solution of the problem

$$\begin{cases} \mathrm{div}\mathbf{V} = -\xi' U_n^{[J^*]}, \quad x \in \Omega_{H/2,H}, \\ \mathbf{V}|_{\partial\Omega_{H/2,H}} = \mathbf{a}, \end{cases} \tag{5}$$

where $\Omega_{H/2,H} = \{x \in \mathbf{R}^n : |x'| < \varphi(x_n), \ x_n \in (H/2, H)\} \bigcup \Omega_0$.

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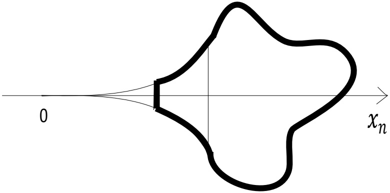
where $\Omega_{H/2,H} = \{x \in \mathbf{R}^n : \ |x'| < \varphi(x_n),\ x_n \in (H/2, H)\} \bigcup \Omega_0$.



We define $\mathbf{V}(x', x_n) = 0$ for $x_n \leq H/2$, while for $x_n \geq H/2$ the function $\mathbf{V}$ is the solution of the problem

$$\begin{cases} \mathrm{div}\mathbf{V} = -\xi' U_n^{[J^*]}, \quad x \in \Omega_{H/2,H}, \\ \mathbf{V}|_{\partial\Omega_{H/2,H}} = \mathbf{a}, \end{cases} \tag{5}$$

where $\Omega_{H/2,H} = \{x \in \mathbf{R}^n : \; |x'| < \varphi(x_n), \; x_n \in (H/2, H)\} \bigcup \Omega_0$.

Solvability condition

$$-\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*]} = \int\limits_{\partial\Omega_{H/2,H}} \mathbf{a} \cdot \mathbf{n}$$

is satisfied, since

$$-\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*]} \, dy' \, dy_n = -\int\limits_{H/2}^{H} \xi' \int\limits_{\sigma} U_n^{[J^*]} \, dy' \, dy_n = F \int\limits_{H/2}^{H} \xi' dy_n = -F$$

$$\begin{aligned}\mathbf{u}(x',x_n) &= \xi(x_n)\mathbf{U}^{[J^*]}(x'/x_n^\lambda, x_n) + \mathbf{V}(x',x_n) + \widehat{\mathbf{U}}(x',x_n),\\ p(x',x_n) &= \xi(x_n)P^{[J^*]}(x'/x_n^\lambda, x_n) + \widehat{P}(x',x_n).\end{aligned} \tag{6}$$

And $(\widehat{\mathbf{U}}, \widehat{P})$ is the solution to the problem

$$\begin{cases} -\nu\Delta\widehat{\mathbf{U}} + \nabla\widehat{P} = & \mathbf{f} + \xi\mathbf{H}_{J^*} + \nu\Delta\mathbf{V} + \nu(2\nabla\xi\cdot\nabla\mathbf{U}^{[J^*]} \\ & +\Delta\xi\mathbf{U}^{[J^*]}) - \nabla\xi\cdot P^{[J^*]}, \quad x\in\Omega, \\ \operatorname{div}\widehat{\mathbf{U}} = 0, & \\ \widehat{\mathbf{U}}|_{\partial\Omega} = 0, & \end{cases} \tag{7}$$

satisfying zero flux condition; where $J^* \in \mathbf{N}$ is a certain number which ensures that $\mathbf{H}_{J^*}$ belongs to L^2.

Theorem 1

Let $\mathbf{f} \in L^2(\Omega)$, $\mathbf{a} \in W^{1/2,2}(\partial\Omega_{H/2,H})$ be given functions and $\operatorname{supp} \mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_{H/2,H} \cap \partial\Omega$. Then the problem (1) admits at least one weak solution $\mathbf{u} \in W^{1,2}_{loc}(\Omega) \cap W^{1,2}(\Omega_{H/2,H})$, which can be represented as a sum (6), where $\mathbf{U}^{[J^*-1]}$ is the constructed asymptotic expansion, $\mathbf{V}$ is a solution to the problem (5), and $\widehat{\mathbf{U}}$ is a weak solution to the problem (7). Moreover, the following estimate

$$\|\mathbf{u} - \xi\mathbf{U}^{[J^*-1]}\|_{W^{1,2}(\Omega)} \leq C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega_{H/2,H})}\right) \quad (8)$$

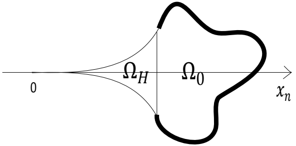
holds.

We consider time-periodic Stokes problem:

$$\begin{cases} \mathbf{u}_t(x,t) - \nu\Delta\mathbf{u}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t), \quad x \in \Omega, \\ \operatorname{div}\mathbf{u}(x,t) = 0, \quad x \in \Omega, \\ \mathbf{u}(x,t)|_{\partial\Omega} = \mathbf{a}(x,t), \\ \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), \end{cases} \tag{9}$$

$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,ds = F(t) \not\equiv 0, \tag{10}$$

where $\operatorname{supp}\mathbf{a} \in \partial\Omega_0 \cap \partial\Omega$.



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$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,ds = F(t) \not\equiv 0, \tag{10}$$

where $\operatorname{supp}\mathbf{a} \in \partial\Omega_0 \cap \partial\Omega$, $\nu > 0$ is the constant kinematic viscosity, and we assume that $\mathbf{f}(x,0) = \mathbf{f}(x,2\pi)$, $F(0) = F(2\pi)$.

$$F(t) + \int\limits_{\partial\Omega\cap\partial\Omega_0} \mathbf{a}(x,t)\cdot\mathbf{n}\,dx = 0.$$

Definition 1

Let $\mathbf{f} \in L^2(0, 2\pi; L^2(\Omega))$, $\mathbf{a}$, $\mathbf{a}_t \in L^2(0, 2\pi; W^{1/2,2}(\partial\Omega_{H/2,H}))$. By a weak solution of problem (9) we understand a solenoidal time-periodic vector field $\mathbf{u} \in L^2(0, 2\pi; W^{1,2}_{loc}(\Omega) \cap W^{1,2}(\Omega_{H/2,H}))$ with $\mathbf{u}_t \in L^2(0, 2\pi; L^2_{loc}(\Omega) \cap L^2(\Omega_{H/2,H}))$ satisfying the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{a}$ and the integral identity

$$\int\limits_0^{2\pi}\int\limits_\Omega \mathbf{u}_\tau(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau + \nu\int\limits_0^{2\pi}\int\limits_\Omega \nabla\mathbf{u}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)\,dxd\tau$$

$$= \int\limits_0^{2\pi}\int\limits_\Omega \mathbf{f}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau,$$

for every time-periodic solenoidal $\boldsymbol{\eta} \in L^2\left(0, 2\pi; C_0^\infty(\Omega)\right)$.

Let us consider the homogeneous problem (9), (10) with zero boundary condition in the domain Ω_H.

$$(x', x_n, t) \longrightarrow \left(\frac{x'}{x_n^{\lambda}}, x_n, t\right) := (y', y_n, t).$$

$$\begin{cases} \mathbf{u}'_t - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p = 0, \quad y \in \Pi, \\ u_{n,t} - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D}p = 0, \quad y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D}u_n = 0, \quad y \in \Pi, \\ \mathbf{u}|_{\partial\Pi} = 0, \\ \mathbf{u}(y', y_n, 0) = \mathbf{u}(y', y_n, 2\pi), \end{cases}$$

where $\Delta' = \nabla' \cdot \nabla'$, $\nabla' = (\partial_1, ..., \partial_{n-1})$, $\mathfrak{D} = \partial_n - \lambda y_n^{-1} y' \cdot \nabla'$, $\mathbf{u}' = (u_1, ..., u_{n-1})$, $\partial_k = \frac{\partial}{\partial y_k}$.

We look for the solution $(\mathbf{U}_0, P_0)$ in the form

$$P_0(y', y_n, t) = q_0(y_n)g_0(t) + Q_0(y', y_n, t),$$
$$\mathbf{U}_0(y', y_n, t) = (\mathbf{U}_0'(y', y_n, t), U_{n,0}(y', y_n, t))$$

with

$$U_{n,0}(y', y_n, t) = y_n^{2\lambda}\partial_n q_0(y_n)\Phi(y', t).$$

($(\mathbf{U}_0, P_0)$ is not the exact solution!)

Substituting $(\mathbf{U}_0, P_0)$ into equations as $y_n \to 0$, we get

$$\begin{cases} -\nu\Delta'\mathbf{U}_0' + \nabla' y_n^{\lambda} Q_0 = 0, \quad y' \in \omega, \\ \mathrm{div}'\mathbf{U}_0' = G_0(y', y_n, t), \quad y' \in \omega, \\ \mathbf{U}_0'|_{\partial\omega} = 0, \end{cases} \qquad (*)$$

$$\begin{cases} \nu\Delta'\Phi(y', t) = g_0(t), \quad y' \in \omega, \\ \Phi(y', t)|_{\partial\omega} = 0, \end{cases}$$

where $\omega = \left\{y' \in \mathbf{R}^{n-1} : \, |y'| < \gamma_0\right\}$.

$$G_0(y', y_n, t) = -y_n^{\lambda}\mathfrak{D}(y_n^{2\lambda}\partial_n q_0(y_n)\Phi(y', t)).$$

$$\begin{cases} \nu\Delta'\Phi(y',t) = g_0(t), & y' \in \omega, \\ \Phi(y',t)|_{\partial\omega} = 0, \end{cases}$$

The solution $\Phi(y',t)$ has the form

$$\Phi(y',t) = g_0(t)\varphi(y'),$$

where function φ is the solution to the Poisson equation

$$\begin{cases} \nu\Delta'\varphi(y') = 1, & y' \in \omega \\ \varphi(y')|_{\partial\omega} = 0. \end{cases}$$

$$\varphi(y') = \frac{1}{2\nu(n-1)}\left(|y'|^2 - \gamma_0^2\right).$$

$$\begin{cases} -\nu\Delta' \mathbf{U}_0' + \nabla' y_n^{\lambda} Q_0 = 0, \quad y' \in \omega, \\ \operatorname{div}' \mathbf{U}_0' = G_0(y', y_n, t), \quad y' \in \omega, \\ \mathbf{U}_0'|_{\partial\omega} = 0, \end{cases} \qquad (*)$$

$$G_0(y', y_n, t) = -y_n^{\lambda}\mathfrak{D}(y_n^{2\lambda}\partial_n q_0(y_n)\Phi(y', t)),$$

where $\mathfrak{D} = \partial_n - \lambda y_n^{-1} y' \cdot \nabla'$.

$$(*) \overset{\text{is solvable}}{\iff} \int\limits_{\omega} G_0 \, dy' = 0.$$

$$(*) \overset{\text{is solvable}}{\Longleftrightarrow} \int\limits_{\omega} G_0 \, dy' = 0$$

$$\Rightarrow -y_n^{\lambda}\partial_n \left[y_n^{2\lambda}\partial_n q_0(y_n) \right] g_0(t) - \lambda(n-1) y_n^{3\lambda-1}\partial_n q_0(y_n) g_0(t) = 0$$

$$\Rightarrow \partial_n \left[y_n^{\lambda(n+1)}\partial_n q_0(y_n) \right] = 0$$

$$\Rightarrow q_0(y_n) = C_1 y_n^{1-\lambda(n+1)} + C_2$$

$$U_{n,0}(y', y_n, t) = y_n^{2\lambda}\partial_n q_0(y_n)\Phi(y', t)$$

$$(*) \overset{\text{is solvable}}{\Longleftrightarrow} \int\limits_{\omega} G_0 \, dy' = 0$$

$$\Rightarrow -y_n^{\lambda}\partial_n \left[y_n^{2\lambda}\partial_n q_0(y_n)\right] g_0(t) - \lambda(n-1)y_n^{3\lambda-1}\partial_n q_0(y_n)g_0(t) = 0$$

$$\Rightarrow \partial_n \left[y_n^{\lambda(n+1)}\partial_n q_0(y_n)\right] = 0$$

$$\Rightarrow q_0(y_n) = C_1 y_n^{1-\lambda(n+1)} + C_2$$

$$\overset{C_1=1}{\Rightarrow} U_{n,0}(y', y_n, t) = [1-\lambda(n+1)]y_n^{-\lambda(n-1)} g_0(t)\varphi(y')$$

Moreover, since

$$\int\limits_{\sigma(h)} \mathbf{u}_0(x,t)\cdot\mathbf{n}\,dx' = \kappa_0(1-\lambda(n+1))g_0(t),$$

taking $g_0(t) = F(t)\,[\kappa_0(1-\lambda(n+1))]^{-1}$, we satisfy the flux condition (10); where $\kappa_0 = \int\limits_{\omega}\varphi(y')\,dy' \neq 0$.

Comparing the power exponents of y_n in (*) and G_0 we conclude that functions $\mathbf{U}_0'(y',y_n,t)$, $Q_0(y',y_n,t)$ have to be taken in the form

$$\begin{cases} \mathbf{U}_0'(y',y_n,t) = y_n^{-\lambda(n-2)-1}\mathscr{U}_0'(y',t), \\ Q_0(y',y_n,t) = y_n^{-\lambda(n-1)-1}\mathscr{Q}_0(y',t). \end{cases}$$

Therefore, we can rewrite:

$$\begin{cases} -\nu\Delta'\mathscr{U}_0' + \nabla'\mathscr{Q}_0 = 0, \quad y' \in \omega, \\ \mathrm{div}'\mathscr{U}_0' = \mathscr{G}_0(y', t), \quad y' \in \omega, \\ \mathscr{U}_0'|_{\partial\omega} = 0, \end{cases}$$

where

$$\mathscr{G}_0(y', t) = \frac{\lambda}{\kappa_0} F(t) A(y', \nabla')\varphi(y'),$$

the operator A is given by

$$A(y', \nabla') = n - 1 + y' \cdot \nabla'.$$

Finally,

$$\begin{cases} \mathbf{U}_0'\left(\frac{x'}{x_n^\lambda}, x_n, t\right) = x_n^{-\lambda(n-2)-1}\mathscr{U}_0'\left(\frac{x'}{x_n^\lambda}, t\right), \\ Q_0\left(\frac{x'}{x_n^\lambda}, x_n, t\right) = x_n^{-\lambda(n-1)-1}\mathscr{Q}_0\left(\frac{x'}{x_n^\lambda}, t\right), \end{cases}$$

$$U_{n,0}\left(\frac{x'}{x_n^\lambda}, x_n, t\right) = \frac{F(t)}{\kappa_0} x_n^{-\lambda(n-1)}\varphi\left(\frac{x'}{x_n^\lambda}\right).$$

The discrepancies $\mathbf{H}_0'(y', y_n, t)$, $H_{n,0}(y', y_n, t)$ left by functions $\mathbf{U}_0$, Q_0 have the form

$$\begin{cases} \mathbf{H}_0'(y', y_n, t) = & \nu\mathfrak{D}^2\mathbf{U}_0'(y', y_n, t) - \mathbf{U}_{0,t}'(y', y_n, t) \\ = & y_n^{-\lambda(n-2)-3}\widehat{\mathscr{F}}_0'(y', t) + y_n^{-\lambda(n-2)-1}\widetilde{\mathscr{F}}_0'(y', t), \\ H_{n,0}(y', y_n, t) = & \nu\mathfrak{D}^2 U_{n,0}(y', y_n, t) - U_{n,0,t}(y', y_n, t) \\ & -\mathfrak{D} Q_0(y', y_n, t) \\ = & y_n^{-\lambda(n-1)-2}\widehat{\mathscr{F}}_{n,0}(y', t) + y_n^{-\lambda(n-1)}\widetilde{\mathscr{F}}_{n,0}(y', t). \end{cases}$$

The right-hand sides split into two: $\widetilde{\mathscr{F}}$ denotes the discrepancies appearing due to the terms with the derivative with respect to time t, $\widehat{\mathscr{F}}$ - the rest of the terms.

$$\begin{cases} \mathbf{u}'_t - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p = y_n^{-\lambda(n-2)-3}\widehat{\mathscr{F}}'_0(y',t), & y \in \Pi, \\ u_{n,t} - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D}p = y_n^{-\lambda(n-1)-2}\widehat{\mathscr{F}}_{n,0}(y',t), & y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D}u_n = 0, \quad y \in \Pi, \\ \mathbf{u}|_{\partial\Pi} = 0, \quad \mathbf{u}(y', y_n, 0) = \mathbf{u}(y', y_n, 2\pi). \end{cases}$$

We will satisfy these equations in main, if we put

$$P_1(y', y_n, t) = g_1(t)y_n^{-1-\lambda(n-1)} + y_n^{-3-\lambda(n-3)}\mathscr{Q}_1(y', t),$$

$$\mathbf{U}'_1(y', y_n, t) = y_n^{-3-\lambda(n-4)}\mathscr{U}'_1(y', t),$$

$$U_{n,1}(y', y_n, t) = y_n^{-2-\lambda(n-3)}\mathscr{U}_{n,1}(y', t),$$

where

$$\mathscr{U}_{n,1}(y', t) = g_1(t)(1 + \lambda(n-1))\varphi(y') + \mathscr{U}^*_{n,1}(y', t).$$

$\mathscr{U}^*_{n,1}$ satisfies the equations

$$\begin{cases} -\nu\Delta'\mathscr{U}^*_{n,1} = \widehat{\mathscr{F}}_{n,0}, \quad y' \in \omega, \\ \mathscr{U}^*_{n,1}|_{\partial\omega} = 0, \end{cases}$$

$(\mathscr{U}'_1, \mathscr{Q}_1)$ is the solution to

$$\begin{cases} -\nu\Delta'\mathscr{U}'_1 + \nabla'\mathscr{Q}_1 = \widehat{\mathscr{F}}'_0, \quad y' \in \omega, \\ \mathrm{div}'\mathscr{U}'_1 = [\lambda A(y', \nabla') - 2(\lambda - 1)]\mathscr{U}_{n,1}, \quad y' \in \omega, \\ \mathscr{U}'_1|_{\partial\omega} = 0, \end{cases}$$

and the function g_1 is uniquely determined from the solvability condition

$$\int\limits_{\omega} \left[\lambda A(y', \nabla') - 2(\lambda - 1)\right] \mathscr{U}_{n,1}(y')\, dy' = 0.$$

$\mathscr{U}^*_{n,1}$ satisfies the equations

$$\begin{cases} -\nu\Delta'\mathscr{U}^*_{n,1} = \widehat{\mathscr{F}}_{n,0}, \quad y' \in \omega, \\ \mathscr{U}^*_{n,1}|_{\partial\omega} = 0, \end{cases}$$

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and the function g_1 is uniquely determined from the solvability condition ,i.e.

$$g_1(t) = -\frac{1}{\kappa_0\,[1 + \lambda(n-1)]}\int\limits_{\omega} \mathscr{U}^*_{n,1}\, dy'.$$

The discrepancies $\mathbf{H}_1'(y', y_n, t)$, $H_{n,1}(y', y_n, t)$ left by functions $\mathbf{U}_1$, P_1 can be written in the form

$$\begin{cases} \mathbf{H}_1'(y', y_n, t) = & y_n^{-\lambda(n-4)-5}\widehat{\mathscr{F}}_2'(y', t) + y_n^{-\lambda(n-2)-1}\widetilde{\mathscr{F}}_1'(y', t) \\ & +y_n^{-\lambda(n-4)-3}\widetilde{\mathscr{F}}_2'(y', t), \\ H_{n,1}(y', y_n, t) = & y_n^{-\lambda(n-3)-4}\widehat{\mathscr{F}}_{n,2}(y', t) + y_n^{-\lambda(n-1)}\widetilde{\mathscr{F}}_{n,1}(y', t) \\ & +y_n^{-\lambda(n-3)-2}\widetilde{\mathscr{F}}_{n,2}(y', t). \end{cases}$$

$$(\mathbf{U}_0, P_0) \quad (\mathbf{U}_1, P_1) \quad (\mathbf{U}_2, P_2) \quad \dots \quad (\mathbf{U}_{J^*}, P_{J^*}) \quad \dots$$
$$(\mathbf{H}'_0, H_{n,0}) \quad (\mathbf{H}'_1, H_{n,1}) \quad (\mathbf{H}'_2, H_{n,2}) \quad \dots \quad (\mathbf{H}'_{J^*}, H_{n,J^*}) \quad \dots$$

We shall keep constructing the functions $(\mathbf{U}_k, P_k)$, $k = 1, 2, \dots$, till the discrepancies $\mathbf{H}'_k$, $H_{n,k}$ will belong to $L^2(\Omega)$.
Up to certain $\widehat{l}$ only $(\widehat{\mathbf{F}}'_k, \widehat{F}_{n,k})$, $k = 0, 2, \dots, \widehat{l}$, will be taken as the new right-hand sides, while others will only appear later. In particular

$$\widehat{l} = \min\left\{ l \in \mathbf{N} : l \geq \frac{1}{\lambda - 1} - 1 \right\} =$$

$$\begin{cases} \frac{1}{\lambda-1} - 1, & \text{when } \lambda = \frac{N+1}{N},\ N = 1, 2, \dots, \\ \left[\frac{1}{\lambda-1}\right], & \text{when } \lambda \neq \frac{N+1}{N},\ N = 1, 2, \dots . \end{cases}$$

$$\mathbf{k} \leq \widehat{\mathbf{l}}$$

$$0 \quad \widehat{\mathscr{F}}_1 \quad \widehat{\mathscr{F}}_2 \quad \dots \quad \widehat{\mathscr{F}}_{\widehat{l}}$$

$$\mathbf{k} > \widehat{\mathbf{l}}$$

$\lambda = \frac{\mathbf{N+1}}{\mathbf{N}}$	$\lambda \neq \frac{\mathbf{N+1}}{\mathbf{N}}$
$\widehat{\mathscr{F}}_k + \widetilde{\mathscr{F}}_{k-\widehat{l}}$	$\widetilde{\mathscr{F}}_1$
$\widehat{\mathscr{F}}_{k+1} + \widetilde{\mathscr{F}}_{k+1-\widehat{l}}$	$\widehat{\mathscr{F}}_{\widehat{l}+1}$
$\widehat{\mathscr{F}}_{k+2} + \widetilde{\mathscr{F}}_{k+2-\widehat{l}}$	$\widetilde{\mathscr{F}}_2$
...	...
$\widehat{\mathscr{F}}_{J^*} + \widetilde{\mathscr{F}}_{J^*-\widehat{l}}$	$\widehat{\ }$or$\widetilde{\ }$

Case: $\lambda = \frac{N+1}{N}$

In this case the new right-hand sides are of the form

$$\mathbf{F}'_k(y', y_n, t) = y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)} \mathscr{F}'_k(y', t) =$$

$$y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)} \begin{cases} \widehat{\mathscr{F}}'_k(y', t), & k \leq \widehat{l}, \\ \widehat{\mathscr{F}}'_k(y', t) + \widetilde{\mathscr{F}}_{k-\widehat{l}-1}(y', t), & k > \widehat{l}, \end{cases}$$

$$F_{n,k}(y', y_n, t) = y_n^{-\lambda(n-1)-2+2k(\lambda-1)} \mathscr{F}_{n,k} =$$

$$y_n^{-\lambda(n-1)-2+2k(\lambda-1)} \begin{cases} \widehat{\mathscr{F}}_{n,k}(y', t), & k \leq \widehat{l}, \\ \widehat{\mathscr{F}}'_{n,k}(y', t) + \widetilde{\mathscr{F}}_{n,k-\widehat{l}-1}(y', t), & k > \widehat{l}, \end{cases}$$

$k = 0, 1, 2, \ldots$

$$\boldsymbol{\mu \longrightarrow \mu + 2(\lambda - 1)}$$

$$\begin{cases} \mathbf{U}'^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y', t), \\ U_n^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y', t), \\ P^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} \Big(g_k(t) y_n^{-\lambda(n+1-2k)+1-2k} (1 \\ & + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\ & + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y', t) \Big), \end{cases}$$

$$\mathscr{U}_{n,k}(y', t) = g_k(t)(-\lambda(n+1-2k) + 1 - 2k)\varphi(y') + \mathscr{U}^*_{n,k}(y', t),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

$$\mu \longrightarrow \mu + 2(\lambda - 1)$$

$$\begin{cases} \mathbf{U}'^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y', t), \\ U_n^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y', t), \\ P^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} \Big(g_k(t) y_n^{-\lambda(n+1-2k)+1-2k} (1 \\ & + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\ & + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y', t) \Big), \end{cases}$$

$$\mathscr{U}_{n,k}(y', t) = g_k(t)(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathscr{U}^*_{n,k}(y', t),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

$$\mathscr{U}'_k(y',t) \quad \mathscr{Q}_k(y',t)$$

$$\begin{cases} -\nu\Delta'\mathscr{U}'_k + \nabla'\mathscr{Q}_k = \mathscr{F}'_{k-1}, \quad y' \in \omega, \\ \mathrm{div}'\mathscr{U}'_k = [\lambda A(y',\nabla') - 2k(\lambda-1)]\,\mathscr{U}_{n,k}, \quad y' \in \omega, \\ \mathscr{U}'_k|_{\partial\omega} = 0, \end{cases}$$

$$\mathscr{U}_{n,k}(y',t) = g_k(t)(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathscr{U}^*_{n,k}(y',t).$$

$$\boldsymbol{\mu \longrightarrow \mu + 2(\lambda - 1)}$$

$$\begin{cases} \mathbf{U}'^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y', t), \\ U_n^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y', t), \\ P^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} \Big(g_k(t) y_n^{-\lambda(n+1-2k)+1-2k} (1 \\ & + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\ & + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y', t) \Big), \end{cases}$$

$$\mathscr{U}_{n,k}(y', t) = g_k(t)(-\lambda(n+1-2k) + 1 - 2k)\varphi(y') + \mathscr{U}^*_{n,k}(y', t),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

$$\boldsymbol{\mu \longrightarrow \mu + 2(\lambda - 1)}$$

$$\begin{cases} \mathbf{U}'^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y', t), \\ U_n^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y', t), \\ P^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} \Big(g_k(t) y_n^{-\lambda(n+1-2k)+1-2k} (1 \\ & + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\ & + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y', t) \Big), \end{cases}$$

$$\mathscr{U}_{n,k}(y', t) = g_k(t)(-\lambda(n+1-2k) + 1 - 2k)\varphi(y') + \mathscr{U}^*_{n,k}(y', t),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

$$\delta_k^{\bar{k}}$$

$$\begin{cases} \mathbf{U}_{\bar{k}}(y', y_n, t) = \mathbf{U}_k(y', y_n, t), \\ P_{\bar{k}}(y', y_n, t) = g_{\bar{k}}(t) \ln\, y_n + y_n^{\lambda(2\bar{k}-n+1)-1-2\bar{k}} \mathcal{Q}_{\bar{k}}(y', t), \end{cases}$$

$$\mu \longrightarrow \mu + 2(\lambda - 1)$$

$$\begin{cases} \mathbf{U}'^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \mathscr{U}'_k(y', t), \\ U_n^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathscr{U}_{n,k}(y', t), \\ P^{[J]}(y', y_n, t) = & \sum_{k=0}^{J} \Big(g_k(t) y_n^{-\lambda(n+1-2k)+1-2k} (1 \\ & + \delta_k^{\bar{k}} (y_n^{\lambda(n+1-2k)-1+2k} \ln\, y_n - 1)) \\ & + y_n^{-\lambda(n-1-2k)-1-2k} \mathscr{Q}_k(y', t) \Big), \end{cases}$$

$$\mathscr{U}_{n,k}(y', t) = g_k(t)(-\lambda(n+1-2k) + 1 - 2k)\varphi(y') + \mathscr{U}^*_{n,k}(y', t),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

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$$\mathscr{U}_{n,k}(y', t) = g_k(t)(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathscr{U}^*_{n,k}(y', t),$$

here $\delta_k^{\bar{k}}$ is Kronecker's delta.

$$\mathscr{U}^*_{n,k}(y', t)$$

$$\begin{cases} \nu \Delta' \mathscr{U}^*_{n,k} = \mathscr{F}_{n,k-1}, \quad y' \in \omega \\ \mathscr{U}^*_{n,k}|_{\partial\omega} = 0. \end{cases}$$

One can see that, in particular,

$$J^* \geq \min\left\{ J \in \mathbf{N} : J > \frac{1}{4}\left[(n+2)\hat{l} + 2n + 5\right]\right\}$$

corresponds to the case $\lambda = \frac{N+1}{N}$, $N = 1, 2, ...$, and

$$J^* = \widetilde{J} + \widehat{J},$$

-to $\lambda \neq \frac{N+1}{N}$, $N = 1, 2, ...$, where

$$\widetilde{J} = \min\left\{ J \in \mathbf{N} : J > \frac{1}{4}\left[\frac{n-2}{\lambda - 1} + n + 3\right]\right\},$$

$$\widehat{J} = \min\left\{ J \in \mathbf{N} : J > \frac{1}{4}\left[\frac{n+2}{\lambda - 1} + n + 3\right]\right\}.$$

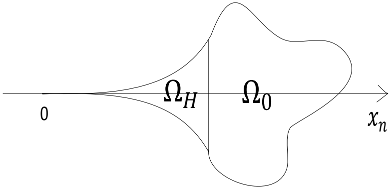
If we want to construct the asymptotic expansion up to the order J we have to assume that the flux $F(t)$ satisfies the following **regularity conditions**

$$F \in W^{J+1,2}(0, 2\pi).$$

Since the flux $F(t)$ is the integral of the normal component of the boundary value $\mathbf{a}(x, t)$ over $\partial\Omega$, we have to assume the following regularity conditions:

$$\frac{\partial^l \mathbf{a}}{\partial t^l} \in L^2(0, 2\pi; W^{1/2,2}(\partial\Omega)), \;\; l = 0, 1, 2, ..., J + 1.$$

Let us consider the problem (9), (10) in the domain Ω.



Let $\xi \in C^\infty[0,\infty)$ be some nonnegative function

$$\xi(x_n) = \begin{cases} 1, & x_n \leq H/2, \\ 0, & x_n \geq H. \end{cases}$$

Then we can look for the solution $(\mathbf{u}, p)$ to problem (9), (10) in the form

$$\mathbf{u}(x', x_n, t) = \xi(x_n)\mathbf{U}^{[J^*]}\left(\frac{x'}{x_n^\lambda}, x_n, t\right) + \mathbf{V}(x', x_n, t) + \widehat{\mathbf{U}}(x', x_n, t),$$

$$p(x', x_n, t) = \xi(x_n)P^{[J^*]}\left(\frac{x'}{x_n^\lambda}, x_n, t\right) + \widehat{P}(x', x_n, t),$$

where $(\mathbf{U}^{[J^*]}, P^{[J^*]})$ is the asymptotical decomposition.
We define $\mathbf{V}(x', x_n, t) = 0$ for $x_n \le H/2$, while for $x_n \ge H/2$ the function $\mathbf{V}$ is the solution of the problem

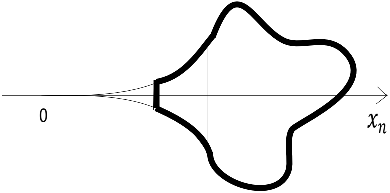
$$\begin{cases} \mathrm{div}\mathbf{V} = -\xi' U_n^{[J^*]}, & x \in \Omega_{H/2,H}, \\ \mathbf{V}|_{\partial\Omega_{H/2,H}} = \mathbf{a}, \end{cases}$$

where $\Omega_{H/2,H} = \{x \in \mathbf{R}^n : |x'| < \varphi(x_n), x_n \in (H/2, H)\} \bigcup \Omega_0$.

We define $\mathbf{V}(x', x_n, t) = 0$ for $x_n \leq H/2$, while for $x_n \geq H/2$ the function $\mathbf{V}$ is the solution of the problem

$$\begin{cases} \mathrm{div}\mathbf{V} = -\xi' U_n^{[J^*]}, \quad x \in \Omega_{H/2,H}, \\ \mathbf{V}|_{\partial\Omega_{H/2,H}} = \mathbf{a}, \end{cases}$$

where $\Omega_{H/2,H} = \{x \in \mathbf{R}^n : \, |x'| < \varphi(x_n), \, x_n \in (H/2, H)\} \bigcup \Omega_0$.



We define $\mathbf{V}(x', x_n, t) = 0$ for $x_n \leq H/2$, while for $x_n \geq H/2$ the function $\mathbf{V}$ is the solution of the problem

$$\begin{cases} \operatorname{div}\mathbf{V} = -\xi' U_n^{[J^*]}, \quad x \in \Omega_{H/2,H}, \\ \mathbf{V}|_{\partial\Omega_{H/2,H}} = \mathbf{a}, \end{cases}$$

where $\Omega_{H/2,H} = \{x \in \mathbf{R}^n : \ |x'| < \varphi(x_n),\ x_n \in (H/2, H)\} \bigcup \Omega_0$.

Solvability condition

$$-\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*]} = \int\limits_{\partial\Omega_{H/2,H}} \mathbf{a} \cdot \mathbf{n}$$

is satisfied, since

$$-\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*]}\, dy'\, dy_n = -\int\limits_{H/2}^{H} \xi' \int\limits_{\sigma} U_n^{[J^*]}\, dy'\, dy_n = F(t) \int\limits_{H/2}^{H} \xi' dy_n = -F($$

$$\begin{aligned} \mathbf{u}(x', x_n, t) &= \xi(x_n)\mathbf{U}^{[J^*]}(x'/x_n^{\lambda}, x_n, t) + \mathbf{V}(x', x_n, t) + \widehat{\mathbf{U}}(x', x_n, t), \\ p(x', x_n, t) &= \xi(x_n)P^{[J^*]}(x'/x_n^{\lambda}, x_n, t) + \widehat{P}(x', x_n, t). \end{aligned} \tag{11}$$

And $(\widehat{\mathbf{U}}, \widehat{P})$ is the solution to the problem

$$\begin{cases} \widehat{\mathbf{U}}_t - \nu\Delta\widehat{\mathbf{U}} + \nabla\widehat{P} = & \mathbf{f} + \xi\mathbf{H}_{J^*} - \mathbf{V}_t + \nu\Delta\mathbf{V} + \nu(2\nabla\xi\cdot\nabla\mathbf{U}^{[J^*]} \\ & +\Delta\xi\mathbf{U}^{[J^*]}) - \nabla\xi\cdot P^{[J^*]}, \quad x\in\Omega, \\ \operatorname{div}\widehat{\mathbf{U}} = 0, & \\ \widehat{\mathbf{U}}|_{\partial\Omega} = 0, & \widehat{\mathbf{U}}(x,0) = \widehat{\mathbf{U}}(x,2\pi), \end{cases}$$

satisfying zero flux condition; where $J^* \in \mathbf{N}$ is a certain number which ensures that $\mathbf{H}_{J^*}$ belongs to $L^2(\Omega)$.

Theorem 2

Let $\mathbf{f} \in L^2(0,2\pi; L^2(\Omega))$, $\mathbf{a}$, $\mathbf{a}_t \in L^2(0,2\pi; W^{1/2,2}(\partial\Omega_{H/2,H}))$ be given time-periodic functions, $\operatorname{supp}\mathbf{a} \in \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_{H/2,H} \cap \partial\Omega$ and there hold the regularity conditions . Then the problem (9) admits at least one time-periodic weak solution $\mathbf{u} \in L^2(0,2\pi;$ $W^{1,2}_{loc}(\Omega) \cap W^{1,2}(\Omega_{H/2,H}))$, $\mathbf{u}_t \in L^2(0,2\pi; L^2_{loc}(\Omega) \cap L^2(\Omega_{H/2,H}))$ which can be represented as the sum (11) and there holds

$$\begin{aligned}
&\sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot,t) - \xi(\cdot)\mathbf{U}^{[J^*-1]}(\cdot,t)\|^2_{W^{1,2}(\Omega)} \\
&+\|\mathbf{u} - \xi\mathbf{U}^{[J^*-1]}\|^2_{L^2(0,2\pi;W^{1,2}(\Omega))} + \|\mathbf{u}_t - \xi\mathbf{U}_t^{[J^*-1]}\|^2_{L^2(0,2\pi;L^2(\Omega))} \\
&\le C\Big(\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(\Omega))} + \|\mathbf{a}\|^2_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))} \\
&+ \sum_{k=1}^{J^*+1} \Big\|\frac{\partial^k \mathbf{a}}{\partial t^k}\Big\|^2_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}\Big).
\end{aligned}$$

We consider the initial boundary value problem for the Stokes system in domains Ω

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & x \in \Omega, \\ \operatorname{div}\mathbf{u} = 0, & x \in \Omega, \\ \mathbf{u}\big|_{\partial\Omega} = \mathbf{a}, & x \in \partial\Omega, \\ \mathbf{u}(x,0) = \mathbf{b}(x), & x \in \Omega. \end{cases} \tag{12}$$

We assume that $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega$ and that the flux of $\mathbf{a}$ is nonzero, i.e.,

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, ds = -F(t). \tag{13}$$

We consider the initial boundary value problem for the Stokes system in domains Ω

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, \qquad x \in \Omega, \\ \operatorname{div}\mathbf{u} &= 0, \qquad x \in \Omega, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{a}, \qquad x \in \partial\Omega, \\ \mathbf{u}(x,0) &= \mathbf{b}(x), \quad x \in \Omega. \end{cases} \tag{12}$$

And we look for a solution $\mathbf{u}$ of problem (12) satisfying the flux condition, so that the following necessary condition

$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,ds + \int\limits_{\Omega} \mathbf{a}\cdot\mathbf{n}\,dx = 0$$

holds for any $h \in (0, H]$.

Further, we suppose that the initial velocity $\mathbf{b}$ is represented as a sum

$$\mathbf{b}(x) = \zeta(x_n)\mathbf{u}_{sing}(x) + \mathbf{u}_0(x),$$

where ζ is a smooth cut-off function with $\zeta(t) = 1$ for $t \leq H/2$ and $\zeta(t) = 0$ for $t \geq H$,

$$\mathbf{u}_{sing}(x) = \left(x_n^{-\lambda(n-2)-1}\mathbf{u}_s'(x'x_n^{-\lambda}),\ x_n^{-\lambda(n-1)}u_{n,s}(x'x_n^{-\lambda})\right),$$

and $\mathbf{u}_0 \in W^{1,2}(\Omega)$, $\mathbf{u}_s', u_{n,s} \in \mathring{W}^{1,2}(\omega)$, $\omega = \{y' \in \mathbf{R}^{n-1} : |y'| < \gamma_0\}$, $y' = x'x_n^{-\lambda}$. Moreover, the initial velocity $\mathbf{b}$ and the boundary value $\mathbf{a}$ have to satisfy the necessary compatibility conditions

$$\operatorname{div}\mathbf{b}(x) = 0, \quad \mathbf{b}(x)|_{\partial\Omega} = \mathbf{u}_0(x)|_{\partial\Omega} = \mathbf{a}(x,0). \tag{13}$$

The singular part $\mathbf{u}_{sing}$ of the initial condition $\mathbf{b}$ is assumed to be solenoidal and it is "responsible" for the flux in the cusp point O at the time moment $t = 0$:

$$\operatorname{div} \mathbf{u}_{sing} = 0, \quad \int\limits_{\sigma(h)} \mathbf{u}_{sing} \cdot \mathbf{n} \, ds = F(0) \quad \forall h \in (0, H/2). \tag{14}$$

In coordinates y this flux condition takes the form

$$\int\limits_{\omega} u_{n,s}(y) \, dy = F(0).$$

Note that the conditions (13) and (14) imply $\operatorname{div} \mathbf{u}_0 = 0$ in $\Omega_{H/2}$.

Definition 3

By a weak solution of the problem (12) we understand a solenoidal vector field $\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega \setminus \Omega_h))$ with $\mathbf{u}_t \in L^2(0, T; L^2(\Omega \setminus \Omega_h))$, $\forall h \in (0, H)$, satisfying the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{a}$, the initial condition $\mathbf{u}(x, 0) = \mathbf{u}_0(x) + \mathbf{u}_s(x)$ and the integral identity

$$\int_0^T \int_\Omega \mathbf{u}_\tau(x,\tau) \cdot \boldsymbol{\eta}(x,\tau)\, dxd\tau + \nu \int_0^T \int_\Omega \nabla \mathbf{u}(x,\tau) \cdot \nabla \boldsymbol{\eta}(x,\tau)\, dxd\tau = \int_0^T \int_\Omega \mathbf{f}(x,\tau) \cdot \boldsymbol{\eta}(x,\tau)\, dxd\tau,$$

for every solenoidal $\boldsymbol{\eta} \in L^2(0, T; C_0^\infty(\Omega))$.

The formal asymptotic expansion near the point O in the case $\lambda = \frac{N+1}{N}$ has the form

$$\mathbf{U}^{[J]}\left(\frac{x'}{x_n^\lambda}, x_n, t, \tau\right) = \mathbf{U}^{O,[J]}\left(\frac{x'}{x_n^\lambda}, x_n, t\right) + \mathbf{U}^{B,[J]}\left(\frac{x'}{x_n^\lambda}, x_n, \tau\right),$$

$$P^{[J]}\left(\frac{x'}{x_n^\lambda}, x_n, t, \tau\right) = P^{O,[J]}\left(\frac{x'}{x_n^\lambda}, x_n, t\right) + P^{B,[J]}\left(\frac{x'}{x_n^\lambda}, x_n, \tau\right).$$

The pair $(\mathbf{U}^{O,[J]}, P^{O,[J]})$ is an approximate solution of the steady Stokes problems ("outer" part of the asymptotic expansion); here the "slow" time variable t plays the role of a parameter and the initial condition is not satisfied in general case. The pair $(\mathbf{U}^{B,[J]}, P^{B,[J]})$ is the boundary layer corrector (the inner part of the asymptotic expansion) which compensate the discrepancy in the initial condition and exponentially vanishes as $\tau \to \infty$. Note that the fast time variable $\tau = \frac{t}{x_n^{2\lambda}}$ in our case depends on x_n.

The approximate solution $(\mathbf{U}^{O,[J]}, P^{O,[J]})$ is constructed in the same way as for the time-periodic problem: Consider the problem (12) with homogeneous boundary conditions in the domain Ω_H (remind that $\mathbf{u}|_{\partial\Omega_H \cap \partial\Omega} = 0$). We rewrite the problem (12) in coordinates $y' = x' x_n^{-\lambda}, y_n = x_n, t = t$:

$$\begin{cases} \mathbf{u}'_t - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p &= 0, \quad y \in \Pi, \\ u_{n,t} - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D} p &= 0, \quad y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D} u_n &= 0, \quad y \in \Pi, \\ \mathbf{u} &= 0, \quad y \in \partial\Pi, \\ \mathbf{u}(y', y_n, 0) &= 0, \quad y \in \Pi, \end{cases}$$

where $\Pi = \{y \in \mathbf{R}^n : \ |y'| < \gamma_0, \ y_n \in (0, H)\}$, $\Delta' = \nabla' \cdot \nabla'$, $\mathfrak{D} = \partial_n - \lambda y_n^{-1} y' \cdot \nabla'$, $\mathbf{u}' = (u_1, ..., u_{n-1})$, $\partial_k = \frac{\partial}{\partial y_k}$, $k = 1, ..., n$, $\nabla' = (\partial_1, ..., \partial_{n-1})$, $\mathrm{div}' \, \mathbf{u}' = \nabla' \cdot \mathbf{u}'$.

The approximate solution $(\mathbf{U}^{O,[J]}, P^{O,[J]})$ is constructed in the same way as for time-periodic problem:

$$\mathbf{U}'^{O,[J]}(y,t) = y_n^{-\lambda(n-2)-1}\mathscr{U}'_0(y',t) + \sum_{k=1}^{J} y_n^{-\lambda(n-2-2k)-2k-1}\mathscr{U}'_k(y',t),$$

$$U_n^{O,[J]}(y,t) = \frac{F(t)}{\kappa_0} y_n^{-\lambda(n-1)}\varphi(y') + \sum_{k=1}^{J} y_n^{-\lambda(n-1-2k)-2k}\mathscr{U}_{n,k}(y',t),$$

$$P^{O,[J]}(y,t) = \frac{F(t)}{\kappa_0(1-\lambda(n+1))} y_n^{1-\lambda(n+1)} + y_n^{-\lambda(n-1)-1}\mathscr{Q}_0(y',t)$$
$$+ \sum_{k=1}^{J}\left[y_n^{-\lambda(n+1-2k)+1-2k} g_k(t) + y_n^{-\lambda(n-1-2k)-1-2k}\mathscr{Q}_k(y',t)\right],$$

where $J \in \mathbf{N}$.

In general, the vector function $(\mathbf{U}_0', U_{n,0})$ does not satisfy the initial condition. Therefore, we have to construct a boundary layer near the point $t = 0$ (on this step we satisfy only the singular part of the initial condition). Rewriting (12) in Ω_H in new coordinates

$$y' = x' x_n^{-\lambda}, y_n = x_n, \tau = t/x_n^{2\lambda},$$

we get

$$\begin{cases} y_n^{-2\lambda}\mathbf{u}'_\tau - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}_b^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p & = & 0, & y \in \Pi, \\ y_n^{-2\lambda}u_{n,\tau} - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}_b^2)u_n + \mathfrak{D}_b p & = & 0, & y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D}_b u_n & = & 0, & y \in \Pi, \\ \mathbf{u} & = & 0, & y \in \partial\Pi, \\ \mathbf{u}(y,0) = \mathbf{u}_s(y) & - & \mathbf{U}_0(y,0), & y \in \Pi, \end{cases}$$

where $\mathfrak{D}_b = \partial_n - \lambda y_n^{-1} y' \cdot \nabla' - 2\lambda y_n^{-1}\tau\partial_\tau$.

We look for the solution $(\mathbf{U}_0^b, P_0^b)$ in the form

$$\mathbf{U}_0^b(y', y_n, \tau) = \left(\mathbf{U}_0'^b(y', y_n, \tau), U_{n,0}^b(y', y_n, \tau)\right),$$
$$P_0^b(y', y_n, \tau) = y_n^{1-\lambda(n+1)} g_0^b(\tau) + Q_0^b(y', y_n, \tau),$$

where

$$\mathbf{U}_0'^b(y', y_n, \tau) = y_n^{-\lambda(n-2)-1} \mathscr{U}_0'^b(y', \tau),$$
$$Q_0^b(y', y_n, \tau) = y_n^{-\lambda(n-1)-1} \mathscr{Q}_0^b(y', \tau),$$
$$U_{n,0}^b(y', y_n, \tau) = y_n^{-\lambda(n-1)} \mathscr{U}_{n,0}^b(y', \tau),$$

with

$$\mathscr{U}_{n,0}^b(y', \tau) = (1 - \lambda(n+1))\, \Phi_0^b(y', \tau).$$

Substituting $(\mathbf{U}_0^b, P_0^b)$ into equations and selecting the leading as $y_n \to 0$ terms, we get

$$\begin{cases} \mathscr{U}'^b_{0,\tau} - \nu \Delta' \mathscr{U}'^b_0 + \nabla' \mathscr{Q}^b_0 &= 0, & y' \in \omega, \\ \operatorname{div}' \mathscr{U}'^b_0 &= \lambda A_b(y', \tau, \nabla', \partial_\tau) \mathscr{U}^b_{n,0}(y', \tau), & y' \in \omega, \\ \mathscr{U}'^b_0 &= 0, & y' \in \partial\omega, \\ \mathscr{U}'^b_0(y', 0) &= \mathbf{u}'^b_0(y'), & y' \in \omega, \end{cases}$$

where $A_b(y', \tau, \nabla', \partial_\tau) = n - 1 + y' \cdot \nabla' + 2\tau\partial_\tau$. Solvability condition

$$\int_\omega A_b(y', \tau, \nabla', \partial_\tau) \mathscr{U}^b_{n,0}(y', \tau)\, dy' \equiv 0$$

is satisfied automatically, since it is equivalent to

$$(1 - \lambda(n+1)) 2\tau\partial_\tau \int_\omega \Phi^b_0(y', \tau)\, dy' = 0.$$

and

$$\begin{cases} \partial_\tau \Phi_0^b(y',\tau) - \nu\Delta'\Phi_0^b(y',\tau) & := & s_0^b(\tau), & y' \in \omega, \\ \Phi_0^b(y',\tau) & = & 0, & y' \in \partial\omega, \\ \Phi_0^b(y',0) & := & u_{n,0}^b(y'), & y' \in \omega, \\ \int\limits_\omega \Phi_0^b(y',\tau)dy' & = & 0, \end{cases}$$

where $u_{n,0}^b(y') = u_{n,s}(y') - \mathscr{U}_{n,0}(y',0)$ and

$$s_0^b(\tau) = -g_0^b(\tau) - 2\frac{\lambda}{\lambda(n+1)-1}\tau\frac{dg_0^b(\tau)}{d\tau}.$$

By construction there holds the following compatibility condition

$$\int\limits_\omega \Phi_0^b(y',0)dy' = \int\limits_\omega u_{n,0}^b(y')\,dy' = 0.$$

and

$$\begin{cases} \partial_\tau \Phi_0^b(y',\tau) - \nu\Delta'\Phi_0^b(y',\tau) & := & s_0^b(\tau), & y' \in \omega, \\ \Phi_0^b(y',\tau) & = & 0, & y' \in \partial\omega, \\ \Phi_0^b(y',0) & := & u_{n,0}^b(y'), & y' \in \omega, \\ \int\limits_\omega \Phi_0^b(y',\tau)dy' & = & 0, & \end{cases}$$

where $u_{n,0}^b(y') = u_{n,s}(y') - \mathscr{U}_{n,0}(y',0)$ and

$$s_0^b(\tau) = -g_0^b(\tau) - 2\frac{\lambda}{\lambda(n+1)-1}\tau\frac{dg_0^b(\tau)}{d\tau}.$$

By construction there holds the following compatibility condition

$$\int\limits_\omega \Phi_0^b(y',0)dy' = \int\limits_\omega u_{n,0}^b(y')\,dy' = 0.$$

$$s_0^b(\tau) = -g_0^b(\tau) - 2\frac{\lambda}{\lambda(n+1)-1}\tau\frac{dg_0^b(\tau)}{d\tau}.$$

The function g_0^b is the solution to the given ODE and has the form

$$g_0^b(\tau) = \left(-M_0\int\limits_0^\tau s_0^b(t)t^{M_0-1}\,dt + C\right)\tau^{-M_0}, \quad M_0 = \frac{n+1}{2}-\frac{1}{2\lambda} > 1,$$

and we set the constant $C=0$ in order to have finite boundary layer pressure P_0^b at point $\tau=0$. Moreover, there hold the following properties

$$\lim_{\tau\to 0} g_0^b(\tau) = -s_0^b(0), \quad \lim_{\tau\to\infty} g_0^b(\tau) = 0.$$

Functions $\mathbf{U}_0, Q_0, \mathbf{U}_0^b, Q_0^b$ leave in the equations the discrepancies $\mathbf{H}_0'(y', y_n, t, \tau)$, $H_{n,0}(y', y_n, t, \tau)$:

$$\begin{aligned}
\mathbf{H}_0'(y', y_n, t, \tau) = &\quad \nu\mathfrak{D}^2\mathbf{U}_0'(y,t) - \mathbf{U}_{0,t}'(y,t) + \nu\mathfrak{D}_b^2\mathbf{U}_0'^b(y,\tau) \\
= &\quad \mathbf{F}_0'^o(y', y_n, t) + \mathbf{F}_0'^b(y', y_n, \tau), \\
H_{n,0}(y', y_n, t, \tau) = &\quad \nu\mathfrak{D}^2 U_{n,0}(y,t) - U_{n,0,t}(y,t) \\
&\quad -\mathfrak{D}\left({y_n}^{-\lambda(n-2)-1}\mathcal{Q}_0(y', t)\right) \\
&\quad +\nu\mathfrak{D}_b^2 U_{n,0}^b(y', y_n, \tau) - \mathfrak{D}_b\mathcal{Q}_0^b(y', y_n, \tau) \\
= &\quad F_{n,0}^o(y', y_n, t) + F_{n,0}^b(y', y_n, \tau).
\end{aligned}$$

Case: $\lambda = \frac{N+1}{N}$

$$\begin{aligned}
\mathbf{U}'^{B,[J]}(y', y_n, \tau) = &\quad \sum_{k=0}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)} \mathscr{U}'^{b}_{k}(y', \tau), \\
U_n^{B,[J]}(y', y_n, \tau) = &\quad \sum_{k=0}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)} \mathscr{U}^{b}_{n,k}(y', \tau), \\
P^{B,[J]}(y', y_n, \tau) = &\quad \sum_{k=0}^{J} \Big(g_k^b(\tau) y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)} \\
&\quad + y_n^{-\lambda(n-1)-1+2k(\lambda-1)} \mathscr{Q}_k^b(y', \tau) \Big),
\end{aligned}$$

$$\mathscr{U}^{b}_{n,k}(y', \tau) = (-\lambda(n+1-2k)+1-2k) \left(\Phi_k^b(y', \tau) + \mathscr{U}^{\diamond}_{n,k}(y', \tau) \right);$$

Case: $\lambda = \frac{N+1}{N}$

$$
\begin{aligned}
\mathbf{U}'^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)} \mathscr{U}'^{b}_{k}(y', \tau), \\
U_n^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)} \mathscr{U}^{b}_{n,k}(y', \tau), \\
P^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} \Big(g_k^b(\tau) y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)} \\
& + y_n^{-\lambda(n-1)-1+2k(\lambda-1)} \mathscr{Q}_k^b(y', \tau) \Big),
\end{aligned}
$$

$$
\mathscr{U}^{b}_{n,k}(y', \tau) = (-\lambda(n+1-2k)+1-2k) \left(\Phi_k^b(y', \tau) + \mathscr{U}^{\diamond}_{n,k}(y', \tau) \right);
$$

The functions $\mathscr{U}'^{b}_{k}$, $\mathscr{Q}^{b}_{k}$, $k = 1, 2, \ldots$, are solutions to the problems

$$\begin{cases} \mathscr{U}'^{b}_{k,\tau} - \nu \Delta' \mathscr{U}'^{b}_{k} + \nabla' \mathscr{Q}^{b}_{k} = \mathscr{F}'^{b}_{k-1}, \quad y' \in \omega, \\ \operatorname{div}' \mathscr{U}'^{b}_{k} = \left[\lambda A_b(y', \tau, \nabla', \partial_\tau) - 2k(\lambda - 1)\right] \mathscr{U}^{b}_{n,k}, \\ \mathscr{U}'^{b}_{k}|_{\partial\omega} = 0, \quad \mathscr{U}'^{b}_{k}(y', 0) = -\mathscr{U}'_{k}(y', 0), \end{cases}$$

Case: $\lambda = \frac{N+1}{N}$

$$
\begin{aligned}
\mathbf{U}'^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)} \mathscr{U}'^{b}_{k}(y', \tau),\\
U_n^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)} \mathscr{U}^{b}_{n,k}(y', \tau),\\
P^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} \Big(g_k^b(\tau) y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)} \\
& + y_n^{-\lambda(n-1)-1+2k(\lambda-1)} \mathscr{Q}_k^b(y', \tau) \Big),
\end{aligned}
$$

$$
\mathscr{U}^{b}_{n,k}(y', \tau) = (-\lambda(n+1-2k)+1-2k) \left(\Phi_k^b(y', \tau) + \mathscr{U}^{\diamond}_{n,k}(y', \tau) \right);
$$

The functions $\mathscr{U}^{\diamond}_{n,k}$ satisfy the equations

$$\begin{cases} \partial_\tau \mathscr{U}^{\diamond}_{n,k} - \nu \Delta' \mathscr{U}^{\diamond}_{n,k} = (-\lambda(n+1-2k)+1-2k)^{-1} \mathscr{F}^{b}_{n,k-1}, \quad y' \in \omega, \\ \mathscr{U}^{\diamond}_{n,k}|_{\partial\omega} = 0, \quad \mathscr{U}^{\diamond}_{n,k}(y', 0) = 0, \end{cases}$$

Case: $\lambda = \frac{N+1}{N}$

$$
\begin{aligned}
\mathbf{U}'^{B,[J]}(y', y_n, \tau) = \quad & \sum_{k=0}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)} \mathscr{U}'^{b}_{k}(y', \tau), \\
U_n^{B,[J]}(y', y_n, \tau) = \quad & \sum_{k=0}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)} \mathscr{U}^{b}_{n,k}(y', \tau), \\
P^{B,[J]}(y', y_n, \tau) = \quad & \sum_{k=0}^{J} \Big(g_k^b(\tau) y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)} \\
& + y_n^{-\lambda(n-1)-1+2k(\lambda-1)} \mathscr{Q}_k^b(y', \tau) \Big),
\end{aligned}
$$

$$
\mathscr{U}^{b}_{n,k}(y', \tau) = (-\lambda(n+1-2k)+1-2k)\left(\Phi_k^b(y', \tau) + \mathscr{U}^{\diamond}_{n,k}(y', \tau)\right);
$$

The functions (Φ_k^b, s_k^b), $k = 1, 2, \ldots$, are solutions to the inverse problems

$$\begin{cases} \partial_\tau \Phi_k^b(y', \tau) - \nu \Delta' \Phi_k^b(y', \tau) := s_k^b(\tau), \quad y' \in \omega, \\ \Phi_k^b(y', \tau)|_{\partial\omega} = 0, \quad \Phi_k^b(y', 0) = -\mathscr{U}_{n,k}(y', 0), \\ \int\limits_\omega \Phi_k^b(y', \tau)\, dy' = -\int\limits_\omega \mathscr{U}^\diamond_{n,k}(y', \tau)\, dy'. \end{cases}$$

Case: $\lambda = \frac{N+1}{N}$

$$\begin{aligned}
\mathbf{U}'^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)} \mathscr{U}'^{b}_{k}(y', \tau), \\
U_n^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)} \mathscr{U}^{b}_{n,k}(y', \tau), \\
P^{B,[J]}(y', y_n, \tau) = & \sum_{k=0}^{J} \Big(g_k^b(\tau) y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)} \\
& + y_n^{-\lambda(n-1)-1+2k(\lambda-1)} \mathscr{Q}^b_k(y', \tau) \Big),
\end{aligned}$$

$$\mathscr{U}^b_{n,k}(y', \tau) = (-\lambda(n+1-2k)+1-2k) \left(\Phi^b_k(y', \tau) + \mathscr{U}^{\diamond}_{n,k}(y', \tau) \right);$$

Finally, the functions $g_k^b(\tau)$ are solutions to the following ODE

$$s_k^b(\tau) = -g_k^b(\tau) - 2c_k\tau\frac{dg_k^b(\tau)}{d\tau},$$

where $c_k = \dfrac{\lambda}{\lambda(n-1)+1-2(k-1)(\lambda-1)}$ and we find

$$g_k^b(\tau) = \left(-M_k\int_0^\tau s_k^b(t)t^{M_k-1}\,dt\right)\tau^{-M_k}, \quad \text{if } \; M_k > 0,$$

$$g_k^b(\tau) = \left(-M_k\int_\tau^\infty s_k^b(t)t^{M_k-1}\,dt\right)\tau^{-M_k}, \quad \text{if } \; M_k < 0,$$

where

$$M_k = \frac{n+1}{2} - k + \frac{2k-1}{2\lambda} > 3/2 - k.$$

Examining the right-hand sides of these problems we state the loss of one time derivative on each step of the outer asymptotic formula construction. Therefore, in order to ensure the existence of all terms of asymptotic expansion up to the order J we have to assume that the flux $F(t)$ is from the space $W^{J+1,2}(0,T)$. Since the flux $F(t)$ is the integral of the normal component of the boundary value $\mathbf{a}(x,t)$ over $\partial\Omega$, the last requirement imply, the following regularity conditions for $\mathbf{a}$:

$$\frac{\partial^l \mathbf{a}}{\partial t^l} \in L^2(0,T; W^{1/2,2}(\partial\Omega)), \quad l = 0, 1, 2, ..., J+1.$$

The boundary layer construction does not cause any loss of regularity, and it is enough to suppose that $\mathbf{u}_0 \in W^{1,2}(\Omega)$, $\mathbf{u}_s', u_{n,s} \in \mathring{W}^{1,2}(\omega)$. Note that the above regularity conditions remain the same as in the case of the time-periodic Stokes

We look for the solution $(\mathbf{u}, p)$ to problem (12), (13) in the domain Ω in the form

$$\begin{aligned} \mathbf{u}(x', x_n, t) &= \xi(x_n)\mathbf{U}^{[J^*-1]}(x,t) + \widehat{\mathbf{U}}(x', x_n, t), \\ p(x', x_n, t) &= \xi(x_n)P^{[J^*-1]}(x,t) + \widehat{P}(x', x_n, t), \end{aligned} \tag{15}$$

where $\xi \in C^\infty[0,\infty)$ is nonnegative function

$$\xi(x_n) = \begin{cases} 1, & x_n \le H/2, \\ 0, & x_n \ge H. \end{cases}$$

the pair $(\mathbf{U}^{[J^*-1]}, P^{[J^*-1]})$ is the asymptotical decomposition and $(\widehat{\mathbf{U}}, \widehat{P})$ is a solution to the problem...

and $(\widehat{\mathbf{U}}, \widehat{P})$ is a solution to the problem

$$\begin{cases} \widehat{\mathbf{U}}_t(x,t) - \nu\Delta\widehat{\mathbf{U}}(x,t) + \nabla\widehat{P}(x,t) = \mathbf{f}^*(x,t), \quad x \in \Omega, \\ \operatorname{div}\widehat{\mathbf{U}}(x,t) = -\xi'(x_n)U_n^{[J^*-1]}(x,t), \\ \widehat{\mathbf{U}}(x,t)|_{\partial\Omega} = \mathbf{a}(x), \quad \widehat{\mathbf{U}}(x,0) = \mathbf{u}_0(x), \end{cases}$$

with $\mathbf{f}^* = \mathbf{f} + \xi\mathbf{H}_{J^*-1} + \nu(2\nabla\xi\cdot\nabla\mathbf{U}^{O,[J^*-1]} + \Delta\xi\mathbf{U}^{O,[J^*-1]}) - \nabla\xi P^{O,[J^*-1]} + \nu(2\nabla\xi\cdot\nabla\mathbf{U}^{B,[J^*-1]} + \Delta\xi\mathbf{U}^{B,[J^*-1]}) - \nabla\xi P^{B,[J^*-1]}$.

Solvability condition

$$\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*-1]}\,dx + \int\limits_{\partial\Omega_{H/2,H}} \mathbf{a}\cdot\mathbf{n}\,dx = 0$$

is satisfied.

Theorem 3

Let functions $\mathbf{f} \in L^2(0, T; L^2(\Omega))$, $\mathbf{u}_0 \in W^{1,2}(\Omega), \mathbf{u}_s \in \mathring{W}^{1,2}(\omega)$, $\mathbf{a}$, $\dfrac{\partial^k \mathbf{a}}{\partial t^k} \in L^2(0, T; W^{1/2,2}(\partial\Omega_{H/2,H}))$, $k = 1, \dots, J^* + 1$, be given and satisfy the compatibility conditions, $\operatorname{supp} \mathbf{a} \in \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_{H/2,H} \cap \partial\Omega$. Then the problem (12) admits at least one weak solution $\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega \setminus \Omega_h))$, $\mathbf{u}_t \in L^2(0, T; L^2(\Omega \setminus \Omega_h))$ $\forall h \in (0, H)$, which can be represented as the sum (15). Moreover, the following estimate holds

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|\mathbf{u}(\cdot, t) - \xi(\cdot)\mathbf{U}^{[J^*-1]}(\cdot, t)\|^2_{W^{1,2}(\Omega)} \\
&+\|\mathbf{u} - \xi\mathbf{U}^{[J^*-1]}\|^2_{L^2(0,T;W^{1,2}(\Omega))} + \|\mathbf{u}_t - \xi\mathbf{U}_t^{[J^*-1]}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq C\Big(\|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))} + \|\mathbf{a}\|^2_{L^2(0,T;W^{1/2,2}(\partial\Omega_{H/2,H}))} \\
&+ \sum_{k=1}^{J^*+1} \Big\|\frac{\partial^k \mathbf{a}}{\partial t^k}\Big\|^2_{L^2(0,T;W^{1/2,2}(\partial\Omega_{H/2,H}))} + \|\mathbf{u}_0\|^2_{W^{1,2}(\Omega)} + \|\mathbf{u}_s\|^2_{W^{1,2}(\omega)}\Big).
\end{aligned}
$$

The results of this thesis are published in the following papers:

1. A. Eismontaite, K. Pileckas, On singular solutions of time-periodic and steady Stokes problems in a power cusp domain, *Applicable Analysis*, **97**(3) (2018), 415–437.
2. A. Eismontaite, K. Pileckas, On singular solutions of the initial boundary value problem for the Stokes system in a power cusp domain, *Applicable Analysis*, (2018), Advanced online publication: 20 Apr 2018, https://doi.org/10.1080/00036811.2018.1460815.